Quantum Gravity on a Manifold with Boundaries

Schrödinger Evolution and Constraints

J. A. Rosabal

INSTITUTE FOR THEORETICAL AND MATHEMATICAL PHYSICS
Lomonosov Moscow State University

March-9

Based on: 2111.03467

THERE IS PLEASURE IN RECOGNISING OLD THINGS FROM A NEW VIEWPOINT.

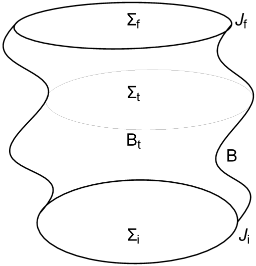
Table of Contents

Motivation

- Hawking's calculation. If a black hole starts in a pure quantum state, why the final state is thermal (mixed state)?
- Unitary *vs* non-unitary evolution.
- Several proposal, none of the is conclusive.
- Information paradox.
- S-matrix in QG.
- Time evolution in QG.

Table of Contents

EH-GYH and Junction Terms



$$S = \frac{1}{16\pi}\int_{M} d^4x\sqrt{-g}\,R + \frac{1}{8\pi}\int_{\Sigma_f-\Sigma_i} d^3y\sqrt{h}\,K - \frac{1}{8\pi}\int_{B} d^3\bar{y}\sqrt{-\bar{h}}\,\bar{K} + \frac{1}{8\pi}\int_{J_f-J_i} d^2z\sqrt{\bar{\sigma}}\,\eta$$

EH-GYH and Junction Terms

$$\frac{1}{8\pi} \int_{J_f - J_i} d^2 z \sqrt{\bar{\sigma}}\, \eta$$

$$\eta = \operatorname{arcsinh}\left(\hat{n}^{(\Sigma)} \cdot \hat{n}^{(B)}\right)$$

G. Hayward and K. Wong, "Boundary Schrodinger equation in quantum geometrodynamics," Phys. Rev. D ***46****, 620-626 (1992).*
S. W. Hawking and C. J. Hunter, "The Gravitational Hamiltonian in the presence of nonorthogonal boundaries," Class. Quant. Grav. ***13****, 2735-2752 (1996)*

EH-GYH and Junction Terms

$$
\begin{aligned}
\delta S = & \frac{1}{16\pi}\int_{M} d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}\right)\delta g^{\mu\nu} \\
& + \frac{1}{16\pi}\int_{\Sigma_f-\Sigma_i} d^3y\sqrt{h}\left(K^{ab} - K\, h^{ab}\right)\delta h_{ab} \\
& - \frac{1}{16\pi}\int_{B} d^3\overline{y}\sqrt{-\overline{h}}\left(\overline{K}^{ab} - \overline{K}\, \overline{h}^{ab}\right)\delta\overline{h}_{ab}.
\end{aligned}
$$

$$
h_{ab} = g_{\mu\nu}\partial_a X^{\mu}_{(\Sigma)}\partial_b X^{\mu}_{(\Sigma)}, \quad \text{and} \quad \overline{h}_{ab} = g_{\mu\nu}\partial_a X^{\mu}_{(B)}\partial_b X^{\mu}_{(B)}
$$

Einstein's equations

$$
R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 0
$$

Table of Contents

ADM actions,and Boundary Conditions

$$g_{\mu\nu} \to (N, N^a, h_{ab}), \quad a = 1, 2, 3$$

$$ds^2 = -N^2dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$$

$$ds^2_{|_{t_{cons}}} = h_{ab}dx^a dx^b$$

$$h_{ab} \to (\Lambda, \Lambda^i, \sigma_{ij}), \quad i = 2, 3$$

$$ds^2_{|_{t_{cons}}} = \Lambda^2 dr^2 + \sigma_{ij}(dx^i + \Lambda^i dr)(dx^j + \Lambda^j dr)$$

ADM actions, and BC

Now, the induced metric on B, takes the form

$$ds^2_{|_{r_{cons}}} = -\Big(N^2 - (\Lambda N^r)^2\Big)dt^2 + \bar{\sigma}_{ij}\Big(dx^i + (\Lambda^i N^r + N^i)dt\Big)\Big(dx^j + (\Lambda^j N^r + N^j)dt\Big)$$

Boundary conditions over B

$$\begin{aligned} N^2 - (\Lambda N^r)^2 &= \bar{N}^2 \\ \Lambda^i N^r + N^i &= \bar{N}^i \end{aligned}$$

$$ds^2_{|_{r_{cons}}} = -\bar{N}^2 dt^2 + \bar{\sigma}_{ij}(dx^i + \bar{N}^i dt)(dx^j + \bar{N}^j dt)$$

ADM actions, and BC

Summarizing

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} \to (N, N^r, N^i, \Lambda, \Lambda^i, \sigma_{ij})$$

$$\text{Over } \Sigma_{i,f} : \left(\Lambda, \Lambda^i, \sigma_{ij}\right)$$

$$\text{Over B} : \left(\overline{N}, \overline{N}^i, \overline{\sigma}_{ij}\right)$$

ADM actions, and BC

Summarizing

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$$g_{\mu\nu} \to (N, N^r, N^i, \Lambda, \Lambda^i, \sigma_{ij})$$

$$\text{Over } \Sigma_{i,f} : \left(\Lambda, \Lambda^i, \sigma_{ij}\right)$$
$$\text{Over B} : \left(\overline{N}, \overline{N}^i, \overline{\sigma}_{ij}\right)$$

$$\begin{aligned} \text{Over B} : \quad N^2 - (\Lambda N^r)^2 &= \overline{N}^2 \\ \Lambda^i N^r + N^i &= \overline{N}^i \end{aligned}$$

ADM action, Hamiltonian and Surface Terms

$$S = \int dt \int_{\Sigma_t} d^3x \Big(\Pi^{ab}\partial_t h_{ab} - N\mathcal{H} - N_a\mathcal{H}^a \Big)$$

$$- \int dt \int_{B_t} dx^2 \sqrt{\overline{\sigma}} \Big(2^{(3)}\hat{r}_a \frac{\Pi^{ab}}{\sqrt{h}} N_b + (8\pi)^{-1} N^{(2)}K$$

$$- (8\pi)^{-1} \frac{\partial_t(\sqrt{\overline{\sigma}}) - \partial_j(\sqrt{\overline{\sigma}}\ \overline{N}^j)}{\sqrt{\overline{\sigma}}}\eta \Big)$$

S. W. Hawking and C. J. Hunter, "The Gravitational Hamiltonian in the presence of nonorthogonal boundaries," Class. Quant. Grav. ***13****, 2735-2752 (1996)*

ADM action, Hamiltonian and Surface Terms

$$\text{Over B}: \quad \begin{aligned} N^2 - (\Lambda N^r)^2 &= \overline{N}^2 \\ \Lambda^i N^r + N^i &= \overline{N}^i \end{aligned}$$

$$\begin{aligned} N_{|_{r_{cons}}} &= \overline{N}\cosh(\eta) \\ \Lambda N^r_{|_{r_{cons}}} &= \overline{N}\sinh(\eta) \end{aligned}$$

$$S_B = -\int dt \int_{B_t} dx^2 \Big(-P\,\overline{N}\,\sinh(\eta) + (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}K\,\overline{N}\,\cosh(\eta) - (8\pi)^{-1}\big(\partial_t(\sqrt{\overline{\sigma}}) - \partial_j(\sqrt{\overline{\sigma}}\,\overline{N}^j)\big)\eta - P_j\overline{N}^j \Big)$$

ADM action, Hamiltonian and Surface Terms

Boundary e.o.m

$$-P\ \overline{N}\cosh(\eta) + (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}K\ \overline{N}\sinh(\eta)$$
$$-(8\pi)^{-1}\left(\partial_t(\sqrt{\overline{\sigma}}) - \partial_j(\sqrt{\overline{\sigma}}\ \overline{N}^j)\right) = 0$$

ADM action, Hamiltonian and Surface Terms

Boundary e.o.m

$$-P\ \overline{N}\ \cosh(\eta) + (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}K\ \overline{N}\ \sinh(\eta)$$
$$-(8\pi)^{-1}\left(\partial_t(\sqrt{\overline{\sigma}}) - \partial_j(\sqrt{\overline{\sigma}}\ \overline{N}^j)\right) = 0$$

Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}R\ g_{\mu\nu} = 0$$

ADM action, Hamiltonian and Surface Terms

Boundary e.o.m

$$-\mathrm{P}\ \overline{\mathrm{N}}\cosh(\eta)+(8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}\mathrm{K}\ \overline{\mathrm{N}}\sinh(\eta)$$
$$-(8\pi)^{-1}\left(\partial_t(\sqrt{\overline{\sigma}})-\partial_j(\sqrt{\overline{\sigma}}\ \overline{\mathrm{N}}^j)\right)=0$$

IT IS AN ON-SHELL IDENTITY

Table of Contents

Schrödinger Evolution

$$\Psi = \int D\mu_1 D\mu_2 e^{iS}$$

$$D\mu_1 = \prod_{t\in[t_i,t_f]} \left[DP\ DP_i\ DP^{ij}\ DN\ DN^a \right]\Big|_{\Sigma_t} \prod_{t\in(t_i,t_f)} \left[D\Lambda\ D\Lambda^i\ D\sigma_{ij} \right]\Big|_{\Sigma_t}$$

$$D\mu_2 = \prod_{t\in[t_i,t_f]} \left[DP\ DP_i\ DP^{ij}\ D\eta \right]\Big|_{B_t} \prod_{t\in(t_i,t_f)} \left[D\Lambda\ D\Lambda^i \right]\Big|_{B_t}$$

We assume that the measure is invariant under translation of N, and N^a, over M; and invariant under translation of η, over B.

Schrödinger Evolution

$$S = \int dt \int_{\Sigma_t} drd^2x \Big(P\partial_t\Lambda + P_i\partial_t\Lambda^i + P^{ij}\partial_t\sigma_{ij} - N\mathcal{H} - N^a\mathcal{H}_a \Big)$$
$$+ \int dt \int_{B_t} dx^2 \Big((8\pi)^{-1}\eta\ \partial_t\sqrt{\overline{\sigma}} - \overline{N}\ \overline{\mathcal{H}} - \overline{N}^j\ \overline{\mathcal{H}}_j \Big)$$

where

$$\begin{aligned} \overline{\mathcal{H}} &= -P\sinh(\eta) + (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}K\cosh(\eta), \\ \overline{\mathcal{H}}_j &= -\big(P_j + (8\pi)^{-1}\partial_j\eta\big) \end{aligned}$$

Schrödinger Evolution

The variation of Ψ, induced by the change of the boundary data $(\Lambda, \Lambda^i, \sigma_{ij})$, on Σ_f, while we push forward this surface, is given by

$$
\begin{aligned}
\delta\Psi = \mathrm{i}\int \mathrm{D}\mu\Bigg[&\int_{\Sigma_f} \mathrm{drd}^2x\Big(\mathrm{P}\delta\Lambda + \mathrm{P}_i\delta\Lambda^i + \mathrm{P}^{ij}\delta\sigma_{ij}\Big)\\
&- \int_{\Sigma_f} \mathrm{drd}^2x\Big(\mathrm{N}\mathcal{H} + \mathrm{N}^a\mathcal{H}_a\Big)\delta\mathrm{t}_f\\
&+ \int_{\mathrm{B}_f} \mathrm{dx}^2(16\pi)^{-1}\eta\ \delta\sqrt{\overline{\sigma}}\\
&- \int_{\mathrm{B}_f} \mathrm{dx}^2\Big(\overline{\mathrm{N}}\ \overline{\mathcal{H}} + \overline{\mathrm{N}}^j\ \overline{\mathcal{H}}_j\Big)\delta\mathrm{t}_f\Bigg]\mathrm{e}^{\mathrm{iS}}
\end{aligned}
$$

Schrödinger Evolution

$$
\begin{aligned}
-\mathrm{i}\frac{\delta}{\delta\Lambda}\int \mathrm{D}\mu e^{\mathrm{i}S} &= \int \mathrm{D}\mu\ \mathrm{P}\ e^{\mathrm{i}S},\\
-\mathrm{i}\frac{\delta}{\delta\Lambda^{i}}\int \mathrm{D}\mu e^{\mathrm{i}S} &= \int \mathrm{D}\mu\ \mathrm{P}_{i}\ e^{\mathrm{i}S},\\
-\mathrm{i}\frac{\delta}{\delta\sigma_{ij}}\int \mathrm{D}\mu e^{\mathrm{i}S} &= \int \mathrm{D}\mu\ \mathrm{P}^{ij}\ e^{\mathrm{i}S}
\end{aligned}
$$

Schrödinger Evolution

Proper time and space

$$\begin{aligned}\delta\tau_f &= N\delta t_f \quad ; \quad \delta\overline{\tau}_f = \overline{N}\delta t_f,\\ \delta\chi^a &= N^a\delta t_f \quad ; \quad \delta\overline{\chi}^j = \overline{N}^j\delta t_f\end{aligned}$$

Schrödinger Evolution

$$i\frac{\partial \Psi}{\partial \tau_f} = \int_{\Sigma_f} drd^2x \int D\mu\, \mathcal{H}\, e^{iS} = \int_{\Sigma_f} drd^2x\, \hat{\mathcal{H}}\Psi,$$

$$i\frac{\partial \Psi}{\partial \chi^a} = \int_{\Sigma_f} drd^2x \int D\mu\, \mathcal{H}_a\, e^{iS} = \int_{\Sigma_f} drd^2x\, \hat{\mathcal{H}}_a\Psi$$

$$i\frac{\partial \Psi}{\partial \overline{\tau}_f} = \int_{B_f} d^2x \int D\mu\, \overline{\mathcal{H}}\, ie^{iS},$$

$$\frac{\partial \Psi}{\partial \overline{\chi}^j} = \int_{B_f} d^2x \int D\mu\, \overline{\mathcal{H}}_j\, e^{iS}$$

Constraints

On the one hand, the functional integral defining the wave function Ψ contains integrals over N, and N^a. The value of the integral should be left unchanged by an infinitesimal translation of these integration variables.

$$\int D\mu \left[\frac{\delta S}{\delta N}\right] e^{iS} = \int D\mu \; \mathcal{H} \; e^{iS} = \hat{\mathcal{H}}\Psi = 0$$

and

$$\int D\mu \left[\frac{\delta S}{\delta N^a}\right] e^{iS} = \int D\mu \; \mathcal{H}_a \; e^{iS} = \hat{\mathcal{H}}_a\Psi = 0$$

Constraints

On the one hand, the functional integral defining the wave function Ψ contains integrals over N, and N^a. The value of the integral should be left unchanged by an infinitesimal translation of these integration variables.

$$\int D\mu \left[\frac{\delta S}{\delta N}\right] e^{iS} = \int D\mu \, \mathcal{H} \, e^{iS} = \hat{\mathcal{H}}\Psi = 0$$

and

$$\int D\mu \left[\frac{\delta S}{\delta N^a}\right] e^{iS} = \int D\mu \, \mathcal{H}_a \, e^{iS} = \hat{\mathcal{H}}_a\Psi = 0$$

$$\frac{\partial \Psi}{\partial \tau_f} = \frac{\partial \Psi}{\partial \chi^a} = 0$$

Constraints

On the other hand, the path integral defining the wave function Ψ contains also and integral over η. The value of the integral should be left unchanged by an infinitesimal translation of this integration variable.

$$\int D\mu \left[\frac{\delta S}{\delta \eta}\right] e^{iS} = 0$$

or

$$\int D\mu \Big[P \cosh(\eta) - (8\pi)^{-1} \sqrt{\overline{\sigma}}^{(2)} K \sinh(\eta) - (8\pi)^{-1} \sqrt{\overline{\sigma}}^{(2)} \overline{K} \Big] e^{iS} = 0$$

Constraints

On the other hand, the path integral defining the wave function Ψ contains also and integral over η. The value of the integral should be left unchanged by an infinitesimal translation of this integration variable.

$$\int D\mu \left[\frac{\delta S}{\delta \eta}\right] e^{iS} = 0$$

or

$$\int D\mu \left[P \cosh(\eta) - (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}K \sinh(\eta) - (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}\overline{K}\right] e^{iS} = 0$$

This constraint equation is just the quantum counterpart of the boundary eom.

Constraints

Specializing this equation at $t = t_f$, it can be written as

BOUNDARY CONSTRAINT

$$i\frac{\delta}{\delta\Lambda}\left\langle \cosh(\eta)\right\rangle + (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}K \left\langle \sinh(\eta)\right\rangle + (8\pi)^{-1}\sqrt{\overline{\sigma}}^{(2)}\overline{K}\ \Psi = 0$$

$$\left\langle ... \right\rangle = \int D\mu(...)e^{iS}$$

Conclusions

- This work has pointed out a subtle fact about classical and quantum gravity on manifolds with boundaries. It turns out that on a manifold with spatial boundaries, some of the ADM degrees of freedom that are independent on the bulk, on the timelike boundary, they are not. This fact leads to a boundary equation of motion derived from the Hamiltonian form of the gravity action.
- The emergence of a classical equation of motion on the boundary in the Hamiltonian formulation of gravity seems to contradict the Lagrangian formulation where no boundary equation arises. However, this is not the case, and no contradiction arises because this classical boundary equation turns out to be an identity when evaluated on a solution of the Einstein's equations

Conclusions

- Quantum mechanically, the situation changes drastically concerning the paradigm in QG on manifolds with boundaries. This new classical equation (identity) becomes a constraint equation on the boundary, similar to the Hamiltonian and the momentum constraints on the bulk. Thus, the time evolution of the wave function now is ruled by a Schrödinger like equation depending only on the degrees of freedom on the junction and by this new constraint equation. This new equation certainly will have consequences when studying the time evolution in QG.

Conclusions

- Operatively, this boundary constraint does not look easy to solve. We do not know yet how to implement this constraint in general. Perhaps for some midisuperspace models, like the spherically symmetric one, with a spatial boundary (the problem over the spatial boundary becomes an ordinary quantum mechanics), one can implement it. Or maybe in some two-dimensional models of gravity, like dilaton gravity. JT gravity, and CGHS model. In these cases, the problem over the spatial boundary would be a one-dimensional quantum mechanics problem. Another setup where we could implement this constraint could be in studying perturbations around a classical background in two or four dimensions.

Open questions

- One of the main remaining questions is whether this new constraint equation will allow for time evolution. For example, could it happen that this constraint restricts the Schrödinger evolution of the wave function on the boundary to those solutions that do not depend on time, in a similar way the Hamiltonian and the momentum constraints do in the bulk. This question will be answered elsewhere.

КОНЕЦ

THE END

FIN